M.S Course, 2001 Entrance Examination

1. (5 점) Let H be a subgroup of a group G and let N be a normal subgroup of G. Show that $HN/N \cong H/H \cap N$.

2. (5 점) Prove or disprove:
   (a) $\mathbb{Z}[x]$ is a principal ideal domain.
   (b) $\mathbb{C}[x]$ is a principal ideal domain.
   Here, $\mathbb{Z}$ is the ring of integers and $\mathbb{C}$ is the field of complex numbers.

3. (5 점) Prove:
   (a) Every finite integral domain is a field.
   (b) Every prime ideal in a finite commutative ring with unity is maximal.

4. (5 점) Construct a field with 16 elements.

5. (6 점) Consider $f \in C^1[0,1]$ having the following properties:
   (i) $f(x) \in [0,1]$ for all $x \in [0,1]$, and
   (ii) $\sup_{x \in [0,1]} |f'(x)| < 1$.
   Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

6. (6 점) Solve the differential equation

$$t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} + y = 0.$$

7. (6 점) Compute the integral

$$\int_{|z|=3} \frac{dz}{(z-1)^{50}(z-2)}dz.$$

8. (7 점) Find the Fourier series of the function $f(x) = x$, $(-\pi \leq x \leq \pi)$.

9. (6 점) Let V_n be the volume of the unit ball $B^n(1)$ in $\mathbb{R}^n$. Show that for all integer $n \geq 1$,

$$V_{n+1} = \gamma_n V_n,$$

where

$$\gamma_n = 2\int_0^1 (1-t^2)^{\frac{n}{2}}\,dt.$$

10. (6 점) Find three geodesics on the surface $\{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\}$.

11. (6 점) Show that a connected metric space X with more than one point is uncountable.

12. (7 점) Let X be the subspace of $\mathbb{R}^2$ given as

$$\left(\bigcup_{n=1}^{\infty} \{1/n\} \times [0,1]\right) \cup (\{0\} \times [0,1]) \cup ([0,1] \times \{0\}).$$

(a) Prove or disprove X is a normal space.
(b) Prove or disprove X is locally-connected.
(c) Prove or disprove X is locally compact.
(d) Prove or disprove X is simply-connected.

13. (10 점) Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a linear operator. Suppose that $\lambda_1, \cdots, \lambda_k$ are distinct eigenvalues of T and $v_1, \cdots, v_k$ are corresponding eigenvectors. Show that $v_1, \cdots, v_k$ are linearly independent over F.

14. (10 점) Show that every finite extension of a field is an algebraic extension.

15. (10 점) Let $S = \{(x, y, z) \in R^3 \mid z = \sqrt{x^2 + y^2},\ 0 \leq z \leq 1\}$ be the given surface. Find the surface integral

$$\int\int_S (1 - z) d\sigma.$$

16. (10 점) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function on $B = \{z \in C \mid |z| \leq 1\}$ such that $|f(z)| \leq 1$ for all $z \in B$. Show that

$$\left|\sum_{n=k}^{\infty} a_n z^n\right| \leq \frac{|z|^k}{1 - |z|}.$$

17. (15 점) Let M be a simply connected surface with constant Gaussian curvature -1 and let T be a regular geodesic triangle with interior angles equal to θ. What is the area of T?

18. (15 점) Let M_0 be a Möbius band. Choose n squares in M_0 as in the picture. Let $M_1, M_2, \ldots, M_n$ be Möbius bands with one squares each as in the picture also. Identify one square of M_0 with one of M_i for each i by a 90 degree rotation and then identifying to form a compact surface X with boundary.

(a) Calculate the number of boundary components of X as a function of n.

(b) Calculate the genus of X as a function of n.