M.S Course, 2000 Entrance Examination, TEST I

1. Let N be a normal subgroup of a group G. Prove that the center of N is also a normal subgroup of G. (5점)

2. Let A be an $n \times n$ real matrix and V be the vector space over $\mathbb{R}$ spanned by $\{I, A, A^2, \ldots\}$, where I is the $n \times n$ identity matrix. Prove that $\dim_{\mathbb{R}} V \leq n$. (5점)

3. Let F be a field and $f(x) \in F[x]$. Show that the ideal $\langle f(x) \rangle$ is maximal if and only if $f(x)$ is irreducible. (5점)

4. Show that every finite field has p^n elements for some prime p and a positive integer n. (5점)

5. Let $S = \{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that every subsequence of it has a subsequence converging to the same limit $x \in \mathbb{R}$. Show that S converges to x. (6점)

6. Let $h : [0,1] \to \mathbb{R}$ be a continuous function such that $h(0) = h(1) = 0$. Define f on $[1, \infty)$ by

$$f(n+x) = \begin{cases} \frac{1}{n} h(nx), & 0 \leq x \leq \frac{1}{n}, \\ 0, & \frac{1}{n} \leq x \leq 1, \end{cases}$$

for $n = 1, 2, \cdots$. Show that f is uniformly continuous on $[1, \infty)$. (6점)

7. Let

$$F(x, y, z) = (e^y \cos x, e^y \sin x, 2 - \cos z).$$

Find the set of points (x, y, z) where the Inverse Function Theorem holds. (6점)

8. Let $P(z) = z^4 - 1$. Compute

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{z^2 P'(z)}{P(z)} dz.$$

(7점)

9. 유클리드 공간에서 곡선 C_1 의 점 p_1 에서 곡률이 1이라 하자. 이 곡선을 2배로 확대한 곡선 C_2 에서 p_1 에 대응되는 점을 p_2 라 하자. 점 p_2 에서 C_2 의 곡률은 얼마인가? (6점)

10. 원기둥면에서 측지선의 종류를 모두 서술하라. (6점)

11. Let X be a compact Hausdorff space and $x_i \in X$, $i = 1, 2, 3, \ldots$ be distinct points of X. (7점)

(1) Is there a continuous function $f : X \to \mathbb{R}$ such that $f(x_i) = i$, $i = 1, 2, 3$ (for three points)?

(2) Is there a continuous function $f : X \to \mathbb{R}$ such that $f(x_i) = i$, $i = 1, 2, 3, \ldots, n, \ldots$ (for infinitely many points)?

12. Define an equivalence relation on $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ by $x \sim y$ if $x = 2^n y$ for $n \in \mathbb{Z}$. Show that the quotient space $\mathbb{R}_+ / \sim$ is homeomorphic to the unit circle S^1. (6점)

1. Prove that there is no simple group of order 48. (10점)

2. Let $\zeta = e^{2\pi i/11}$. (10점)

   (1) Prove that $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension.

   (2) Find $\mathrm{Gal}(\mathbb{Q}(\zeta + \zeta^{-1})/\mathbb{Q})$ up to isomorphism.

3. Use the Picard iteration argument to construct a solution of the differential equation,

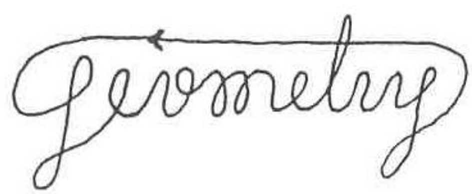
$$\begin{cases} \dfrac{dy}{dt} = ty, \quad t > 0 \\ y(0) = 1. \end{cases}$$

   (10점)

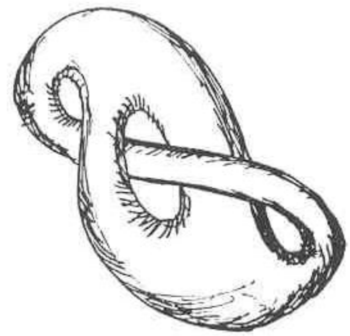
4. Let f be a function continuous in $\mathbb{C}$ and analytic in $\{z \mid \mathrm{Im}\, z \neq 0\}$. Show that f is analytic in $\mathbb{C}$. (10점)

5. 다음과 같은 닫힌 평면곡선과 닫힌 곡면의 전곡률(total curvature)을 구하여라. (15점)

   (1) 곡선 C 의 전곡률은 $\int_C k\, ds$ 으로 정의되는데 여기서 k 는 평면곡선의 곡률로서 음수일 수도 있다. 즉 T, N 이 곡선 C 의 단위접선, 단위법선벡터일 때 $T' = kN$ 인데 N 은 곡선 진행방향의 왼쪽으로 향하는 법선벡터이다.



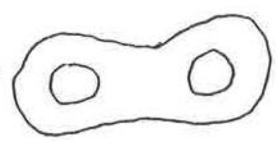
(2) 곡면 S 의 전곡률은 $\int_S K\, dA$ 인데 K 는 가우스 곡률 (Gaussian curvature)이다.



6. Let M_g be an orientable closed surface with genus g and N_k be a non-orientable closed surface with k cross-caps. (15점)

(1) Compute $\pi_1(M_g - D)$ and $\pi_1(N_k - D)$, where D is a small open disk in the surfaces.

(2) Can you identify the following surfaces as $M_g - D$ or $N_k - D$?

$S_1 =$  $S_2 =$