M.S. Course, 1999 Entrance Examination, Test I

1. Prove that every group G of order p^2 is abelian, where p is a prime. Is G cyclic? Justify your answer. (5pts)

2. Let V be a vector space over a field F and let $T : V \to V$ be a linear operator. Let W be the subspace of V spanned by $v, T(v), T^2(v), \cdots$ for some nonzero vector $v \in V$. Prove that if $\dim W = k$, then $\{v, T(v), \cdots, T^{k-1}(v)\}$ is a basis of W. (5pts)

3. Let R be a commutative ring with unity. Prove that every maximal ideal is a prime ideal. (5pts)

4. Let F be a field of p^n elements, where p is a prime and n is an arbitrary positive integer. Prove that a map $\phi : a \mapsto a^p$ for each $a \in F$ is an automorphism of F. (5pts)

5. Find the derivative of the function (6pts)

$$f(x) = \int_0^{2x} \frac{\sin xy^2}{y} dy.$$

6. Solve the following initial value problem: (6pts)

$$\begin{cases} ty'(t) + y(t) = -e^{-t}, 0 < t < \infty \\ y(0) = 0. \end{cases}$$

7. Let $u(x,y) = (x + e^y \sin y, -y + e^x \cos x)$ and S be the unit circle with center at (0,0). Compute the following integral

$$\int_S u(x,y) \cdot n(x,y) ds,$$

where $n(x,y)$ is the unit outer normal vector to S and ds is the line element on S. (6pts)

8. For which values of the real number α does the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\frac{1}{n}\right)^{\alpha}$ converge? (7pts)

9. Let $\gamma = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{4} + y^2 = 1\}$. Compute

$$\int_\gamma \frac{-ydx + (x-1)dy}{(x-1)^2 + y^2},$$

where γ is oriented counter-clockwise with respect to the origin. (6pts)

10. Find the curvature of the parabola $y = x^2$ at the point $(x,y) = (0,0)$ (6pts)

11. Compute the fundamental group of the following spaces. (6pts)

    (a) The three-dimensional Euclidean space removed with the x-axis and the y-axis.

    (b) The above space removed with the line given by $x = 1$ and $y = 1$ where x, y, z are standard coordinate functions.

12. Compute the fundamental group of the following spaces and describe their universal covers and covering maps in a detailed manner. (7pts)

    (a) $X = P^2 \times S^1$ where P^2 is the real projective plane and S^1 is a circle. Describe the universal cover and the covering map.

    (b) X removed with $\{x\} \times S^1$ where x is a point of P^2.

M.S. Course, 1999 Entrance Examination, Test II

1. Let D be a principal ideal domain and $I_1 \subset I_2 \subset \cdots$ be an increasing chain of ideals of D. Show that there is a positive integer n such that $I_n = I_{n+k}$ for all $k \geq 1$. (10pts)

2. Let $\overline{\mathbb{Q}}$ be the subset of $\mathbb{C}$ consisting of all algebraic elements over $\mathbb{Q}$, where $\mathbb{C}$ is the field of complex numbers and $\mathbb{Q}$ is the field of rational numbers. Prove that $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. Is $\mathbb{Q}$ is the unique algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$? Justify your answer. (10pts)

3. If f_n is Riemann integrable on $I = [a, b]$, $(n = 1, 2, \ldots)$ and f_n converges to f uniformly on I as $n \to \infty$, show that f is also Riemann integrable on I. (10pts)

4. Let f be an entire function satisfying $|f(z)| \leq A \log |z| + B|z|^2$ for $|z| \geq R$, where A, B and R are positive constants. Prove that f is a polynomial. (5pts)

5. Compute the total Gaussian curvature of the surface in $\mathbb{R}^3 : x^2 + y^2 - z^2 = 1$. (15pts)

6. A continuous map $f : X \to Y$ is called a *local homeomorphism* if for each $x \in X$ there are neighborhoods U of x and V of $f(x)$ such that $f|_U$ is a homeomorphism of U to V. (15pts)

   (a) Show that a local homeomorphism is an open map.

   (b) Let X and Y be path-connected, Hausdorff, and compact. Show that a local homeomorphism must be a covering map, i.e., show the following statements:

      i. f is onto.

      ii. $f^{-1}(y)$ is finite for each $y \in Y$.

      iii. Each $y \in Y$ has an evenly covered neighborhood.