M.S. Course, 1995 Entrance Examination, Test I

1. How many distinct subgroups does the symmetric group S_3 have? (5pts)

2. If k is a field, show that the polynomial ring $k[x]$ is a PID. (5pts)

3. Find the Galois group of $x^4 - 5x^2 + 6$ over $\mathbb{Q}$. (5pts)

4. Let A be a real $n \times n$ matrix. Show that A is diagonalizable if and only if there is a basis of $\mathbb{R}^n$ consisting of eigen-vectors of A. (5pts)

5. Let $f, g : (-1, 1) \to \mathbb{R}$ be C^∞-functions, and suppose that $f^{(n)}(0) = g^{(n)}(0)$ for $n = 0, 1, 2, \ldots$. Prove or disprove that there is some $\delta > 0$ such that $f(x) = g(x)$ for all $x \in [-\delta, \delta]$. (5pts)

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Define $F_n(x) = \sum_{k=-n}^{n} f(x - 2k\pi)$ on $[-\pi, \pi]$ for $n = 0, 1, 2, \ldots$. Show that the sequence $\{F_n : n = 1, 2, \ldots\}$ converges in $L^1[-\pi, \pi]$. (5pts)

7. For each $n = 1, 2, \ldots$, let

$$f_n(x) = \frac{x}{1 + nx^2} \quad \text{for each } x \in \mathbb{R}.$$

Show that $\{f_n : n = 1, 2, \ldots\}$ converges uniformly to a function f and that the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is correct if $x \neq 0$, but $f'(0) \neq \lim_{n\to\infty} f_n'(0)$. (5pts)

8. Find the general solution of the differential equation $x^2y'' + xy' = 4y$. (5pts)

9. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2 + 2x + 2)^2} dx$. (5pts)

10. 다음 식으로 주어진 곡면의 넓이를 구하라.

$$x = \cos\varphi \cos\theta$$
$$y = \cos\varphi \sin\theta$$
$$x = \sin\varphi$$

단, $\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{3}$, $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$. (7pts)

11. Find the minimum of value of the curvature on the cycloid

$$x = t - \sin t, \; y = 1 - \cos t, \quad 0 \leq t \leq 2\pi$$

in the xy-plane. (7pts)

12. Find $\pi_1(P^2 \times T^2)$, where P^2 is the real projective plane and T^2 is the torus. Find the universal covering space of $P^2 \times T^2$. (7pts)

13. Let X be a compact Hausdorff space. Prove that if x and y are distinct points of X, then there is a continuous real valued function f on X such that $f(x) \neq f(y)$.

M.S. Course, 1995 Entrance Examination, Test II

1. (a) Let g be an element of a finite group G. Show that the number of elements in the conjugate class of g is equal to $|G|/|C_G(g)|$, where $C_G(g)$ is the centralizer of g. (5pts)

   (b) Show that the center of a group of order p^n is nontrivial, where p is a prime and $n > 0$. (5pts)

2. Let F be a finite field.

   (a) Show that the number of elements in F is p^n for some prime p and $n > 0$. (3pts)

   (b) If K is a finite extension of F, show that K is a separable extension of F. (3pts)

   (c) Show that a finite field cannot be algebraically closed. (4pts)

3. If f is an integrable function on $\mathbb{R}$, show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx dx = 0. \quad \text{(10pts)}$$

4. Suppose that f_n is analytic in a domain D in $\mathbb{C}$ and f_n converges to a function uniformly on each compact subset of D. Show that $f(z)$ is analytic in D and also $f_n'(z)$ converges uniformly to $f'(z)$ on each compact subset of D. (10pts)

5. yz-평면 상에서 반지름의 길이가 1이고, 그 중심이 z 축으로부터 2 만큼의 거리에 놓여있는 원을 z 축 둘레로 회전시켰을 때 얻어지는 원환면의 가우스 곡률의 최대값과 최소값을 구하라. (10pts)

6. Let $\mathbb{R}^2$ be a plane. Let $\mathcal{T}$ be a topology on $\mathbb{R}^2$ generated by a basis consisting of sets that equal $\mathbb{R}^2$ subtracted with finitely many complete lines. Show that $(\mathbb{R}, \mathcal{T})$ is compact but not Hausdorff. (10pts)

7. Let $SO(2)$ be the group of 2×2 real matrices A such that AA^T is the identity matrix and $\det A = 1$, where A^T denotes the transpose of A. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

   (a) Are $SO(2)$ and S^1 isomorphic groups? (5pts)

   (b) Are $SO(2)$ and S^1 homeomorphic? (The topology of all 2×2 real matrices is naturally identified with the topology of $\mathbb{R}^4$.) (5pts)