M.S. Course, 1996 Entrance Examination, Test I

1. Show that a subgroup of a cyclic group is cyclic. (5pts)

2. Find all maximal ideals of $\mathbb{C}[x]$, where $\mathbb{C}$ is the field of complex numbers. (5pts)

3. Prove that every algebraic extension of K over F is separable if the characteristic of F is 0. (5pts)

4. Find a real matrix A for which $A^3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$. (5pts)

5. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be given by
$$F(x, y, z) = (x^3 + 2x + e^z - 1, y + z - 1), \quad (x, y, z) \in \mathbb{R}^3.$$
Show that there are an open interval U containing 0 and a differentiable function $\alpha : U \to \mathbb{R}^2$ such that
$$F(x, \alpha(x)) = (0, 0), \quad x \in U,$$
and find the derivative $\alpha'(0)$. (7pts)

6. Find the function $y = f(x)$ which satisfies the differential equation $|x| + |y|y' = 0$ and $f(2) = -1$. (6pts)

7. Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $u(z) \geq 0$ for all $z \in \mathbb{C}$. Prove that u is constant. (6pts)

8. Let C be the curve $x = t - \sin t$, $y = \cos \frac{t}{2}$, $0 \leq t \leq \pi$. Evaluate
$$\int_C y dx + (x + 2y^2) dy. \quad \text{(6pts)}$$

9. Let C be a curve defined by
$$C := \{(x, y) \in \mathbb{R}^2 : x^4 + y^4 = 1\},$$
and let κ be its curvature. Compute the integral $\int_C \kappa$. (6pts)

10. Let S be a surface defined by
$$S := \{(x, y, z) \in \mathbb{R}^3 : x^6 + y^6 + z^6 = 1\}.$$
Compute the integral $\int_C \kappa$, where κ is the Gaussian curvature of S. (6pts)

11. Let us define an equivalence relation on the real line $\mathbf{R}$ (with standard topology) given by $x \sim y$ if $x = 2^n y$ for $n \in \mathbf{Z}$.

(a) Show that $\mathbf{R}/\sim$ is a compact space. (3pts)

(b) Show that $\mathbf{R}/\sim$ is not Hausdorff. (4pts)

12. Let T be a Klein bottle with the interior of an imbedded disk removed.

(a) Find an embedding of T in $\mathbf{R}^3$. (3pts)

(b) Find the presentation of the fundamental group of T. (3pts)

M.S. Course, 1996 Entrance Examination, Test II

1. Classify the groups of order 8 up to isomorphism. (10pts)

2. Let p be an odd prime and $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$.

   (a) Show that $f(x)$ is irreducible over $\mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. (4pts)

   (b) Determine the Galois group of $f(x)$ over $\mathbb{Q}$ up to isomorphism. (6pts)

3. Let f be a Lebesgue integrable function on $\mathbb{R}$. Show that the function defined by
$$F(x) = \int_{-\infty}^{x} f(t)dt, \quad x \in \mathbb{R}$$
is uniformly continuous on $\mathbb{R}$. Is this true for arbitrary measurable function f on $\mathbb{R}$? What can you say about differentiability of f? (10pts)

4. Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{x^2} dx$. (10pts)

5. Suppose A is a 3×3 orthogonal matrix with $\det A = 1$. Verify that the transform $x \mapsto Ax$ is a rotation with respect to some axis. (15pts)

6. Let K be a subcomplex of the projective plane P homeomorphic to a circle represented as in the figure below. Let P' be another copy of P and K' the corresponding circle in P'. Identify K with K' by a simplicial homeomorphism. Let X be the quotient space $(P \cup P')/\sim$. Find a presentation of the fundamental group of X. (15pts)