M.S. Course, 1997 Entrance Examination, Test I

1. Show that no group of order 1996 is simple. (5pts)

2. Let $A = (a_{ij})$ be an $n \times n$ matrix such that
$$\sum_{j=1}^{n} a_{ij} = 1$$
for all $i = 1, 2, \ldots, n$. Show that A has an eigenvalue 1. (5pts)

3. A commutative ring with 1 having a unique maximal ideal is called a local ring. Prove that a nonzero homomorphic image of a local ring is again a local ring. (5pts)

4. Let F be a field and let $f(x) \in F[x]$ be a non-constant polynomial. Prove that there exist an extension field E of F and an $\alpha \in E$ such that $f(\alpha) = 0$. (5pts)

5. Suppose that a C^2-function $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$f''(x) + f'(x) - e^x f(x) \geq 0, \quad x \in \mathbb{R}.$$
Show that f cannot have a nonnegative maximum unless $f \equiv 0$. (7pts)

6. Let $0 < p < \infty$, $\epsilon > 0$ be given. For every measurable function $f : \mathbb{R} \to \mathbb{R}$, show that
$$\mu\{x \in \mathbb{R} : |f(x)| \geq \epsilon\} \leq \frac{1}{\epsilon^p} \int_{\mathbb{R}} |f(x)|^p d\mu(x),$$
where μ is the Lebesgue measure. (6pts)

7. How many roots of the equation $z^4 - 6z + 3 = 0$ have in the annulus $1 < |z| < 2$? Justify your answer. (6pts)

8. Let $P(z)$ be a polynomial and $C = \{z : |z - a| = r\}$, where $r > 0$. Prove that
$$\int_C P(z) d\bar{z} = -2\pi i r^2 P'(a). \text{ (6pts)}$$

9. 좌표공간의 곡선
$$X(t) = \left(\cos \frac{3}{5}t, \sin \frac{3}{5}t, \frac{4}{5}t\right) \quad (0 \leq t \leq 2\pi)$$
의 각 점에서의 곡률 $\kappa(t)$와 꼬임률(torsion) $\tau(t)$를 구하고, 적분값
$$\int_X \kappa \quad \text{와} \quad \int_X \tau$$
를 셈하라. (6pts)

10. 영역 $\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$에서 정의된 곡면

$$X(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

의 넓이를 구하라. (6pts)

11. Prove the tube lemma:
Consider the product space $X \times Y$ where Y is compact. If N is an open subset of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $\{x_0\} \times Y$ where W is a neighborhood of x_0 in X.

12. Let X be the open cube in the euclidean 3-space $\mathbb{R}^3$ given by $|x| < 2$, $|y| < 2$, $|z| < 2$ removed with two disjoint lines l_1 given by $y = z = 0$ and l_2 given by $x = 0$ and $y = 1$. Compute the fundamental group of X.

1. Let G be a finite group and let P be its Sylow p-subgroup. Prove that $N_G(N_G(P)) = N_G(P)$, where $N_G(P)$ is the normalizer of P in G. (10pts)

2. Let $F = \mathbb{F}_q$ be the field of q elements, where q is a prime power. Prove :
$$\prod_{\alpha \in F^\times} \alpha = -1$$
where $F^\times$ denotes the set of all non-zero elements of F. (10pts)

3. For $x \in \mathbb{R}$ and $n = 1, 2, \ldots,$ define
$$f_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{(2k-1)!} x^{2k-1}. \quad \text{(10pts)}$$
   (a) Show that the equation $f_n(x) = 0$ has a unique real zero, say α_n, in the interval $[\frac{3}{4}\pi, \frac{5}{4}\pi]$ for sufficiently large n.
   (b) Show that $\lim_{n\to\infty} \alpha_n = \pi$.

4. Find a one to one holomorphic mapping from the domain $\{z : |z+1| > 1, \ |z+2| < 2\}$ onto $\{z : |z| < 1\}$. (10pts)

5. 좌표공간에서 식
$$X(u, v) = (\cos u, \sin u, v) \quad (0 \le u \le 2\pi, \quad 0 \le v \le 2\pi)$$
으로 주어진 곡면 X의 각 점에서의 가우스 곡률과 평균곡률을 각각 K, H로 두었을 때, 각 점에서 K, H의 값을 구하고, 적분값
$$\int_X K \quad \text{와} \quad \int_X H$$
를 구하라. (15pts)

6. Let X be a compact metric space, and Y a complete metric space. (15pts)
   (a) Show that a map $f : X \to Y$ is continuous if and only if f is uniformly continuous.
   (b) Let $E \subset X$ be a dense subset. Show that $f : E \to Y$ is uniformly continuous if and only if it admits a continuous extension $\tilde{f} : X \to Y$ (i.e. $\tilde{f}|_E = f$)
   (c) Find an example of a compact metric space X and a dense subset E and a continuous function $f : E \to \mathbb{R}$ which does not extend to a continuous function on X.