M.S. Course, 1998 Entrance Examination, Test I

1. Find all conjugacy classes of S_3, the symmetric group on three letters. (5 pts)

2. Let G_1 be a group of order 126. Prove that there exists a group G_2 of order 18 which is a homomorphic image of G_1. (5 pts)

3. Let n be a positive integer not equal to 1. Prove that n is a prime if and only if $(n-1)! \equiv -1 \pmod{n}$. (5pts)

4. Let V be a vector space over a field F such that V admits a basis consisting of finite vectors. Explain briefly why any other basis of V contains the same number of vectors. (You don't have to give a proof. But definitions and statements should be accurate.) (5pts)

5. Let $f : [a,b] \to \mathbf{R}$ be a continuously differential function satisfying $f(0) = 0$. Show that
$$\int_a^b |f(x)|^2 dx \le \frac{(b-a)^2}{2} \int_a^b |f'(x)|^2 dx. \text{ (6pts)}$$

6. Solve $\dot{\mathbf{x}}(t) = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \mathbf{x}(t)$ where $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. (6pts)

7. Let $f : [0,1] \to \mathbf{R}$ be a continuous function and let $\int_0^1 f(x)\varphi(x)dx = 0$ for every $\varphi \in C[0,1]$. Show that $f \equiv 0$. (6pts)

8. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely and uniformly on any compact subset of the half plane Re $z > 1$. (7pts)

9. Find the Gaussian curvature of surface $z = x^2 + y^2$ in $\mathbf{R}^3$ at the point (1,0,1). (6pts)

10. State Stokes' theorem and divergence theorem. Briefly explain your notations. (6pts)

11. (a) Suppose that $\{U_\alpha : \alpha \in I\}$ is a covering of X by open sets U_α for α in an index set I. Show that $f : X \to Y$ is continuous if and only if $f|_{u_\alpha}$ is continuous for each $\alpha \in I$. (3pts)
    (b) Let X be the union of closed subsets A and B. Let $f : A \to Y$ and $g : B \to Y$ be continuous maps. If $f(x) = g(x)$ for every $x \in A \cap B$, then show that the function $h : X \to Y$ defined by letting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$ is continuous. (3pts)

12. A collection of subsets of a space X is said to be locally finite if each $x \in X$ has an open neighborhood which intersets only finitely many members of the collection.

(a) Give us an example of a locally finite collection of subsets in the real line $\mathbf{R}$ with the standard topology. (2pts)

(b) Prove or disprove: the union of a locally finite collection of compact subsets of X is compact. (2pts)

(c) Prove or disprove: the union of a locally finite collection of closed subsets of X is closed. (2pts)

M.S. Course, 1998 Entrance Examination, Test II

1. Show that every PID is a UFD. (10pts)

2. Let F be a cyclic Galois extension of $\mathbf{Q}$, the field of rational numbers, such that $[F : \mathbf{Q}] = 6$. Show that F is not a splitting field of a polynomial of degree 3 in $\mathbf{Q}[x]$. (10pts)

3. Let f be a continuous function on [-1,1]. Suppose that there exists a constant $0 < C < 1$ satisfying

$$\int_{-1}^{1} \frac{|f(t)|}{|t|^n} dt \leq C^n, (n = 1, 2, 3, \dots).$$

   Then show that $f \equiv 0$. (10pts)

4. (a) Find the residue of $\dfrac{\sqrt{z}}{(z+1)^2}$ at $z = -1$. Here $\sqrt{z}$ is the principal branch. (5pts)

   (b) Compute

$$\int_0^\infty \frac{\sqrt{x}}{(x+1)^2} dx. \text{ (5pts)}$$

5. Let $f : \mathbf{R} \to \mathbf{R}$ be any C^2-function satisfying $f'(x) \geq 0$, $f'(0) = 0$ and $f'(1) = 1$. Let C be the plane curve defined by $y = f(x)$,$0 \leq x \leq 1$. Show that the total curvature $\int_C \kappa$ of the curve C is independent of f. (15pts)

6. Let $\mathbf{P}^2$ be the real projective plane, and A a segment with endpoints attached to $\mathbf{P}^2$ at two distinct points.

   (a) Find a presentation of the fundamental group $\pi_1(\mathbf{P}^2)$. (3pts)

   (b) Show that $\mathbf{P}^2$ is homeomorphic to the quotient space of a standard unit disk in the plane $\mathbf{R}^2$ by the equivalence relation defined by $x \sim y$ if and only if $x = y$ when $\|x\|, \|y\| < 1$ or $x = \pm y$ when $\|x\| = \|y\| = 1$. (4pts)

   (c) Show that $\mathbf{P}^2$ removed with a point is homeomorphic to a Möbius band. (3pts)

   (d) Find a presentation of $\pi_1(\mathbf{P}^2 \cup A)$. (3pts)

   (e) Describe the universal cover of $\mathbf{P}^2 \cup A$. (2pts)