

# 대수학 자격 시험 문제 은행

♣♣♣ !!주의!! ♣♣♣

자격시험 기본문제유형의 모든 문제는 실제 시험에서 문제의 유형을 보존하는 한도내에서 변형될 수 있습니다. 예를 들어 기본문제에서 특정한 군이 사용되었다면 다른 군을 사용하여 같은 질문을 할 수 있고, 다항식이 사용되었다면 다른 다항식으로 대체할 수 있습니다.

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## 제 1 절 Group theory

1. Let  $G$  be a group and  $N$  be a normal subgroup of  $G$ . Prove that if  $N$  and  $G/N$  are solvable, then so is  $G$ .
2. Let  $G$  be a finite group and  $H$  be its subgroup of index  $p$ , where  $p$  is the smallest prime dividing the order of  $G$ . Show that  $H$  is a normal subgroup of  $G$ .
3. Let  $G$ ,  $H$  and  $K$  be finitely generated abelian groups. Show that  $G \times H \simeq G \times K$  implies  $H \simeq K$ .
4. Let  $G$  be a finite group whose order is divisible by a prime  $p$ . Show that  $G$  contains an element of order  $p$ .
5. Prove that every  $p$ -group has non-trivial center.
6. Let  $G$  be a group of order  $pq$ , where  $p, q$  are primes. Prove that  $G$  is abelian if  $p < q$  and  $p \nmid q - 1$ .

7. Let  $G$  be a group and  $C(G)$  be the center of  $G$ . Show that if  $G/C(G)$  is cyclic then  $G$  is abelian.
8. Let  $G$  be a group. Given a subgroup  $H$  of  $G$ , define
 
$$H_G = \bigcap_{g \in G} g^{-1}Hg.$$
  - (a) Prove that if  $K$  is a normal subgroup of  $G$  contained in  $H$ , then  $K$  is a subgroup of  $H_G$ .
  - (b) Let  $G = GL(2, \mathbb{Q})$  and  $H$  be the subgroup of  $G$  consisting of non-singular diagonal matrices. Determine  $H_G$ .
9. Let  $G$  be a finite group and  $\hat{G} = \text{Hom}(G, \mathbb{C}^\times)$  be its character group, i.e. the group of homomorphisms of  $G$  into the multiplicative group of nonzero complex numbers.
  - (a) Find  $\hat{G}$  when  $G = S_3$ .
  - (b) Prove that  $G$  and  $\hat{G}$  are isomorphic if  $G$  is abelian.
10. Let  $GL(n, \mathbb{F}_p)$  be the group of nonsingular  $n \times n$  matrices with entries in  $\mathbb{F}_p$  where  $\mathbb{F}_p$  is the field of  $p$  elements with  $p$  prime.
  - (a) Find the order of  $GL(n, \mathbb{F}_p)$ .
  - (b) Prove that there is a Sylow  $p$ -subgroup  $T$  of  $GL(n, \mathbb{F}_p)$  such that every element of  $T$  is a triangular matrix.
11. Prove that the group defined by generators  $a, b$  and relations  $a^2 = e$ ,  $b^3 = e$ ,  $ba = ab^2$  is isomorphic to  $S_3$ .

## 제 2 절 Rings and modules

1. Prove the following.
  - (a) Every Euclidean domain is a principal ideal domain (PID).
  - (b) Every principal ideal domain (PID) is a unique factorization domain (UFD).
2.
  - (a) Let  $R$  be a commutative ring with unity and  $\mathfrak{a}$  be an ideal of  $R$  contained in every maximal ideal of  $R$ . Show that if  $M$  is a finitely generated  $R$ -module satisfying  $\mathfrak{a}M = M$ , then  $M = 0$
  - (b) Let  $R$  be a local ring and  $\mathfrak{m}$  its maximal ideal. Show that if  $N$  is a submodule of a finitely generated  $R$ -module  $M$  such that  $M = N + \mathfrak{m}M$ , then  $M = N$ .
3. Let  $k[[x]]$  be the formal power series ring over a field  $k$ .
  - (a) Describe the units in  $k[[x]]$ .
  - (b) Show that  $k[[x]]$  is a local ring.
  - (c) Show that  $k[[x]]$  is a PID.
4. Let  $R$  be a commutative ring with unity and  $P$  be a prime ideal of  $R$ .
  - (a) Show that the localization  $R_P$  of  $R$  at  $P$  is a local ring.
  - (b) Show that if  $R$  is a PID, then so is  $R_P$ .
5. Let  $R$  be a commutative ring with unity. Show that
$$\{a \in R \mid a^n = 0 \text{ for some integer } n \geq 1\}$$
is equal to the intersection of all prime ideals of  $R$ .
6. Let  $R$  be a commutative ring with unity. Prove the following.
  - (a) If every prime ideal of  $R$  is finitely generated, then  $R$  is Noetherian.
  - (b) If  $R$  is Noetherian, then every finitely generated  $R$ -module is Noetherian.
7. Prove the following.

- (a) Every unitary  $\mathbb{Q}$ -module is both projective and injective.
- (b) No unitary  $\mathbb{Z}$ -module is both projective and injective.

8. Let  $A, B, C$  be modules over a ring  $R$ . Prove that

$$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$$

as  $R$ -modules.

9. (a) Show that for a prime  $p$ , the cyclotomic polynomial

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible in  $\mathbb{Q}[x]$ .

- (b) Show that the polynomial  $x^3 + 17x + 36$  is irreducible in  $\mathbb{Q}[x]$ .
- (c) Prove that  $x^5 - 16x + 2$  is irreducible over  $\mathbb{Q}$  having three real roots.
- (d) Prove that  $x^6 + x^3 + 1$  is irreducible over  $\mathbb{Q}$ .

10. Let  $R$  be a (not necessarily commutative) ring with unity.

- (a) Let  $M$  be a Noetherian left  $R$ -module and  $f : M \rightarrow M$  be an epimorphism. Show that  $f$  is an isomorphism.
- (b) Let  $R$  be a left Noetherian ring with 1 and  $ab = 1$  in  $R$ . Show that  $ba = 1$ .

11. Let  $R = \mathbb{C}[x, y]$  be the polynomial ring and  $J$  be the ideal generated by  $x^2 - y^3$  and  $x + 2y - 3$ . Find all the maximal ideals of  $R/J$ .

12. Let  $R$  be a commutative ring with 1 and  $P$  be a prime ideal of  $R$ . Let  $S$  be an integral extension ring of  $R$ . Prove that there exists a prime ideal  $Q$  of  $S$  such that  $Q \cap R = P$ .

## 제 3 절 Field theory

1. Show that any finite subgroup of the multiplicative group of a field is cyclic.
2. Show that every finite field extension is algebraic.
3. Let  $\zeta = e^{2\pi i/11}$ . Describe the following Galois groups up to isomorphism:
  - (a)  $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ ;
  - (b)  $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}(\zeta + \zeta^{-1}))$ ;
  - (c)  $\text{Gal}(\mathbb{Q}(\zeta + \zeta^{-1})/\mathbb{Q})$
4. Let  $f(x) = x^3 - x - 2 \in \mathbb{Q}[x]$ .<sup>1</sup>
  - (a) Find the Galois group, up to isomorphism, of  $f(x)$  over  $\mathbb{Q}$ .
  - (b) Find the Galois group, up to isomorphism, of  $f(x)$  over  $\mathbb{Q}(\sqrt{-11})$ .
5. Find the Galois group of  $x^3 + x + 1$  over  $F$  for each of the following cases.<sup>2</sup>
  - (a)  $F = \mathbb{R}$ ,      (b)  $F = \mathbb{C}$ ,      (c)  $F = \mathbb{F}_3$ ,      (d)  $F = \mathbb{F}_5$ .
6. Let  $F$  be a field of characteristic 0. Show that every algebraic extension of  $K$  of  $F$  is separable.
7. Prove that an integral domain which is a finite dimensional vector space over a field is a field.
8. Let  $p$  be a prime and  $n \geq 1$  an integer. Show that  $F$  is a finite field with  $p^n$  elements if and only if  $F$  is a splitting field of  $x^{p^n} - x$  over  $\mathbb{Z}_p$ .
9. Let  $F$  be a finite field of characteristic  $p \neq 0$ . Show that  $F = F^p$ .
10. Given a positive integer  $n$ , construct an extension field  $E$  of  $F$  such that  $\text{Gal}(E/F) = S_n$ .
11. Let  $f(x) = x^5 - 16x + 2 \in \mathbb{Z}[x]$ .

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<sup>1</sup>실제 자격시험에서는 다른 3차 다항식이 주어질 수 있습니다.

<sup>2</sup>실제 자격시험에서는 다른 3차 다항식이 주어질 수 있습니다.

- (a) Prove that  $f(x)$  is irreducible over  $\mathbb{Q}$  having three real roots.
  - (b) Is  $f(x)$  solvable by radicals over  $\mathbb{Q}$ ? Justify your answer.
12. Let  $f(x) = x^6 + x^3 + 1$ .
- (a) Prove that  $f(x)$  is irreducible over  $\mathbb{Q}$
  - (b) Find the Galois group of the splitting field  $F$  of  $f(x)$  over  $\mathbb{Q}$ , up to isomorphism.

## 제 4 절 Linear Algebra

1. Let  $T, S : V \rightarrow V$  be diagonalizable linear operators on a finite dimensional vector space  $V$ . Show that if  $TS = ST$ , then they are simultaneously diagonalizable. (If you like, you may assume furthermore that all the eigenspaces of  $T$  and  $S$  are 1-dimensional.)
2. Let  $V$  be an  $n$ -dimensional vector space over a field  $K$  and let  $T : V \rightarrow V$  be a linear transformation such that  $T^2 = T$ .
  - (a) Find all possible characteristic polynomials and their corresponding minimal polynomials for  $T$ .
  - (b) Prove that  $V = \ker T \oplus \operatorname{im} T$ .
3. Let  $V$  be a finite dimensional vector space over a field  $K$ .
  - (a) Identify  $V$  with  $(V^*)^*$ , where  $V^*$  is the dual space of  $V$ .
  - (b) For a subspace  $W$  of  $V$ , we define

$$W^\perp = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\}.$$

Show that  $(W^\perp)^\perp = W$ .

4. Let  $M$  be a real symmetric  $n \times n$  matrix.

- (a) Prove that every eigenvalue of  $M$  is real.
  - (b) Prove that there exists an orthonormal basis of  $\mathbb{R}^n$  consisting of eigenvectors of  $M$ .
  - (c) Prove that if  $M^k = I_n$  for some positive integer  $k$  then  $M^2 = I_n$ , where  $I_n$  is the identity matrix.
5. Let  $K$  be an algebraically closed field and  $A$  be an  $n \times n$  matrix over  $K$ . Show that there is a nonsingular  $n \times n$  matrix  $B$  over  $K$  such that  $B^{-1}AB$  is upper triangular.
  6. Suppose there is a nondegenerate bilinear form  $(\cdot, \cdot)$  on a vector space  $V$  over a field  $K$ . Show that for each linear functional  $\phi : V \rightarrow K$ , there is a unique  $v \in V$  such that

$$\phi(w) = (v, w) \quad \text{for all } w \in V.$$

7. Let  $A$  be an invertible  $n \times n$  matrix over the complex field  $\mathbb{C}$ . Does  $A$  have a square root, i.e. an  $n \times n$  matrix  $B$  such that  $B^2 = A$ ? Justify your answer.