

## TOPOLOGY PROBLEM BANK FOR MASTER DEGREE STUDENTS

Justify your answers as rigorously as possible.

- (1) Answer the following.
  - (a) Show that  $[0, 1]$  with usual topology is not compact.
  - (b) Given a compact metric space  $X$ , show that for any open covering  $\mathcal{A}$ , there exists  $\delta > 0$  such that for each subset of  $X$  of diameter less than  $\delta$ , there is an element of  $\mathcal{A}$  containing it.
  - (c) Show that if a connected subset  $A \subset X$  is both open and closed, then it is a component of  $X$ .
- (2) Answer the following.
  - (a) Show that if  $Y$  is compact, then the projection  $\pi_1 : X \times Y \rightarrow X$  is a closed map.
  - (b) Let  $f : X \rightarrow Y$ , Let  $Y$  be compact Hausdorff. Then,  $f$  is continuous if and only if the graph of  $f$ ,

$$\text{Graph}_f = \{(x, f(x)) \in X \times Y | x \in X\}$$

is closed in  $X \times Y$ .

- (3) Let  $\mathbb{R}^\omega$  be the countably infinite product of  $\mathbb{R}$  with itself.

$$\mathbb{R}^\omega = \prod_{n \in \mathbb{N}} X_n, \text{ where } X_n = \mathbb{R}$$

Let  $\mathbb{R}^\infty$  be a subset of  $\mathbb{R}^\omega$  consisting of all sequences that are “eventually zero” that is  $(x_1, x_2, \dots)$  such that  $x_i \neq 0$  for only finitely many values of  $i$ . What is the closure of the set  $\mathbb{R}^\infty$  in the product topology of  $\mathbb{R}^\omega$ ?

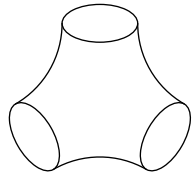
- (4) Answer the following.
  - (a) Show that two sphere and the torus are not homeomorphic.
  - (b) Show that two sphere and the 2 disk are not homotopy equivalent.
- (5) Let  $X$  be a topological space which is obtained from  $S^2$  and  $S^1$  by attaching them along one point.
  - (a) What is  $\pi_1(X)$ ?
  - (b) What is the universal cover of  $X$ ?
- (6) Let  $S_3, S_2$  be genus 3 and 2 closed orientable surfaces, respectively.
  - (a) Prove that there is no 3-fold cover from  $S_3$  to  $S_2$ .

- (b) Show that  $\pi_1(S_2)$  contains a free group  $F_2$  with two generators .
- (7) Calculate  $H_*(\text{Klein bottle}; \mathbb{Z})$ .
- (8) Show that the free group  $F_2$  with two generators has a normal subgroup of index  $n$  for every positive integer  $n$ .
- (9) Answer the following.
  - (a) Let  $X$  be a two dimensional oriented manifold with boundary consisting of two disjoint circles. Suppose two circles are homotopic in  $X$ . Show that  $X$  is homeomorphic to  $S^1 \times [0, 1]$ .
- (10) Let  $X = I \times S^1$  be a topological space.
  - (a) Compute  $H_*(X, \partial X; \mathbb{Z})$  .
  - (b) Describe the generators of  $H_2(X, \partial X; \mathbb{Z})$  geometrically .
- (11) Let  $K$  be a Klein bottle.
  - (a) Find a presentation of the fundamental group  $\pi_1(K)$ .
  - (b) Describe the orientable double cover of  $K$  and the corresponding subgroup of  $\pi_1(K)$ .
  - (c) Describe the deck transformation group of the above covering map.
  - (d) Find all double covering spaces (up to homeomorphism) of  $K$ .
- (12) Let  $X$  be a topological space and suppose that a group  $G$  acts on  $X$  freely with the following condition : For each  $x \in X$ , there exists an open neighborhood  $U$  of  $x$  such that  $gU \cap U = \emptyset$  for all  $g \in G$ . Prove the followings.
  - (a) The quotient map  $q : X \rightarrow X/G$  is a covering map. ( $X/G$  is a quotient space)
  - (b)  $q$  is a normal covering.
- (13) (a) Give a definition of a Hausdorff space.
  - (b) Show that the space  $\mathbb{R}/\sim$  is not a Hausdorff space where  $x \sim y$  if and only if  $x - y$  is rational.
- (14) Classify covering spaces of the torus.
- (15) Answer the following.
  - (a) Show that any continuous map  $f : B^n \rightarrow B^n$ ,  $n > 1$ , where  $B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ , has a fixed point.
  - (b) Show that  $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ ,  $n \geq 2$  is simply connected.
- (16) Show that the projective space  $P^{2n}$  covers no topological space nontrivially.
- (17) Let  $X$  be a bouquet of three circles. Find a three-fold regular cover and a three-fold nonregular cover. State why they are regular or not regular.

- (18) Let  $X, Y, X \cap Y$  be subcomplexes of a simplicial complex  $X \cup Y$ . Prove that

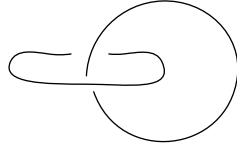
$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y).$$

- (19) Let  $M$  be a closed 2-dimensional manifold and  $\widetilde{M}$  be an orientable double covering of  $M$ .
- Show that such  $\widetilde{M}$  is topologically unique (i.e., all such  $\widetilde{M}$  (if exist) are homeomorphic each other).
  - Determine  $\widetilde{M}$  when  $M = \#_1^n \mathbf{P}^2$  (= connected sum of  $n$  copies of real projective plane).
- (20) (a) Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be a short exact sequence of chain complexes of abelian groups. Compare the homology groups of  $A$  and  $C$  when  $B$  is acyclic.
- (b) Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow 0$  be an exact sequence of chain complexes of abelian groups. Compare the homology groups of  $A$  and  $C$  when  $B$  and  $D$  are acyclic.
- (21) Prove the following statements:
- Any continuous map  $S^n \rightarrow S^n$  without fixed points is homotopic to the antipodal map. ( $S^n$  denotes the unit sphere of dimension  $n$  in  $\mathbb{R}^{n+1}$ .)
  - Let  $G$  be a group of homeomorphisms acting freely on  $S^{2n}$ , i.e., for all  $g \in G$ ,  $gx = x$  for some  $x$  if and only if  $g = 1$ . Then the order of  $G$  is 2.
- (22) In the following problems, all the spaces are assumed to satisfy the conditions for covering space theory.
- Let  $p : \widetilde{Y} \rightarrow Y$  be a covering space with a section, i.e., a map  $s : Y \rightarrow \widetilde{Y}$  such that  $p \circ s = id$ . Show that  $p$  must be a homeomorphism.
  - Let  $X$  be a simply connected space and  $A$  be a connected subset of  $X$ . Show that the quotient space  $Y = X/A$  is also simply connected.
- (23) Let  $P$  be a pair of pants :

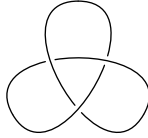


- Compute the homotopy groups  $\pi_*(P)$ .
- Compute the homology groups  $H_*(P; \mathbb{Z})$ .

- (c) Let  $M = P \cup_{\partial} P$  be the surface obtained from the two copies of  $P$  glued along their boundaries with the identity map. Compute  $H_*(M; \mathbb{Z})$ .
- (24) Let  $S^3$  be the standard 3-sphere.
- Describe  $S^3$  as a union of two solid tori. (hint:  $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$  or  $S^3 = \mathbb{R}^3 \cup \{\infty\}$  and consider the complement of a standard solid torus in  $\mathbb{R}^3$ .)
  - Show that  $S^3 - S^1$  has the same homotopy type as a solid torus. Here  $S^1$  is a standard circle, i.e.,  $S^1 = \{(z, 0) \in S^3\}$ .
  - Let  $L$  be the union of two disjoint circle in  $S^3$  as in the right. Compute  $\pi_1(S^3 - L)$  and  $H_*(S^3 - L)$ .



- (25) (a) Compute the homology of the complement of  $S^1$  imbedded in  $\mathbb{R}^3$  as follows. i.e.,  $H_*(\mathbb{R}^3 - S^1; \mathbb{Z})$ .



- (b) Let  $X = S^3 - S^1$  where  $S^3$  is the one-point compactification of  $\mathbb{R}^3$  where  $S^1$  is as above. Is it true that  $X \times S^1$  is homotopy equivalent to a torus?