

다양체 문제은행 (총 20 + 8문제)

1. Let

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

and let

$$f : S^2 \rightarrow \mathbb{R}$$

be defined by $f(x, y, z) = x^2 + 2y^2 + 3z^2$ for $(x, y, z) \in S^2$. Find the critical points of f .

2. Prove or disprove: The standard sphere \mathbf{S}^2 in \mathbb{R}^3 is diffeomorphic to

$$M := \{(x, y, z) \in \mathbb{R}^3 : x^4 + y^6 + z^8 = 1\}.$$

3. Are the following maps orientation preserving?

(a) $X \mapsto -X \quad (X \in \mathbb{R}^n)$

(b) $X \mapsto X/|X|^2 \quad (X \in \mathbb{R}^n - \{0\})$

4. Find all differential 1-forms ξ on the standard unit sphere S^2 such that $r^*\xi = \xi$ for any $r \in SO(3)$.

5. Let $\mathbf{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be oriented as the boundary of the “standard ball” $x^2 + y^2 + z^2 \leq 1$. Compute the integral

$$\int_{\mathbf{S}^2} xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

6. Let ω be a differential 1-form on \mathbb{R} with compact support. Is it true that there exists a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support such that $df = \omega$?

7. Two differential forms on a smooth manifold are said to be *cohomologous* to each other if their difference is an exact form. If ω and ω' are cohomologous differential forms and if ξ and ξ' are cohomologous forms, is $\omega \wedge \xi$ cohomologous to $\omega' \wedge \xi'$?

8. Show that S^2 and P^2 are two-dimensional smooth manifolds.

9. (a) Given a smooth vector field X and a differential form ω on a manifold M , define the Lie derivative $L_X\omega$ of ω with respect to X at $p \in M$.
 (b) Show that $L_X f = X(f)$ whenever $X \in C^\infty(M)$.

10. (a) Show that every closed 1-form on S^2 is exact.

(b) Let

$$\sigma = \frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

in $\mathbb{R}^3 - \{0\}$. Prove that σ is closed but not exact.

11. Use de Rham cohomology to prove that the torus T^2 is not diffeomorphic with S^2 .

12. Is $\{(x, y, z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 1\}$ a smooth submanifold of \mathbb{R}^3 ?

13. Show that the mapping $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$h(x, y, z) = (x \cos xy, z \sin xy, x + z)$$

has an inverse mapping on a neighborhood of $h(1, 0, 1) = (1, 0, 2)$.

14. Let \mathbf{F} be a vector field on \mathbb{R}^2 given by

$$\mathbf{F} = (2x + y, x).$$

(a) Is there a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\nabla f = \mathbf{F}?$$

If so, find f .

(b) Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be a curve in \mathbb{R}^2 given by

$$\gamma(t) = (t, t^2).$$

Compute the following line integral

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{s}.$$

15. Let \mathbf{F} be a vector field on \mathbb{R}^3 given by

$$\mathbf{F} = (x, 2y, 3z),$$

and let S be the sphere of radius r with center at the origin.

(a) What is the $\operatorname{div} \mathbf{F}$?

(b) Compute the following surface integral

$$\int_S \mathbf{F} \cdot \mathbf{n},$$

where \mathbf{n} is the unit outward normal vector field on S .

16. Let M be a C^∞ manifold. Give a proof or a counter-example to the following questions.

(a) Suppose ϕ is a homeomorphism and ϕ is a C^1 map. Is ϕ necessarily a C^1 diffeomorphism?

(b) Suppose ϕ is a C^1 diffeomorphism and ϕ is a C^∞ map. Is ϕ necessarily a C^∞ diffeomorphism?

17. Let $X(x, y) = (x + y, 2x - 3y)$ be a vector field on \mathbb{R}^2 , and let $\omega = 2x dy + y dx$. Compute the Lie derivative $L_X \omega$.

18. Let

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 7 & 3 & 0 \\ 4 & 6 & 2 \end{bmatrix}.$$

Let $f(t) = \det(\exp(tA))$. Compute $f'(0)$.

19. Find the dimensions of the following manifolds. Indicate how you have arrived at your answers, but you do NOT have to prove your assertions.

(a) $SU(n)$

(b) $SO(n)$

20. Let

$$V = (x + y) \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$$

be a vector field on \mathbb{R}^2 . Show that it is complete.

21. Let $X(x, y) = (2x + y, x - y)$ and $Y(x, y) = (x, y)$ be vector fields on \mathbb{R}^2 . Compute the bracket $[X, Y]$.

22. Consider the meromorphic function

$$f(z) = \frac{z^2 + 1}{z^3 + 5z^2 - 7z + 10}$$

as a map from the sphere \mathbf{S}^2 onto itself. Find the degree of this map.

23. Let $\gamma = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{4} + y^2 = 1\}$. Compute

$$\int_{\gamma} \frac{-ydx + (x-1)dy}{(x-1)^2 + y^2},$$

where γ is oriented counter-clockwise with respect to the origin.

24. Let Σ be a compact oriented regular surface without boundary in \mathbb{R}^3 . Suppose that Σ has a nontrivial fundamental group. Does the Gaussian curvature K of Σ vanish at some point?
25. Prove that the tangent bundle TM of a differentiable manifold M is a differentiable manifold which is diffeomorphic, by a fibre preserving diffeomorphism, to the product manifold $M \times \mathbb{R}^n$ iff there exist vector fields $X_1, \dots, X_n \in \mathfrak{X}(M)$ such that $\{X_i(p)\}$ is a basis of T_pM for each $p \in M$.
26. The space of oriented 2 planes (4 dimensional subspaces) of \mathbb{R}^4 is a manifold. Compute the dimension of this space. (You do not have to prove that it is a manifold.)
27. Let $X = y \frac{\partial}{\partial x}$ be a vector field on \mathbb{R}^2 , and let $\omega = y^2 dx - 2xy dy$ be a 1-form on \mathbb{R}^2 . Compute $L_X \omega$.
28. Suppose \mathbf{F} be a vector field on \mathbb{R}^2 given by

$$\mathbf{F} = (2xy + e^y, x^2 + xe^y).$$

Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be a curve in \mathbb{R}^2 given by

$$\gamma(t) = (t, t^2).$$

Compute the following line integral

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{s}.$$