
Department of Mathematical Sciences, SNU
Sample Qualifying exams problems

Analysis

2009

1. Let (f_n) be a sequence of continuous functions converging to f uniformly. If (x_n) is a sequence converging to x , show that $(f_n(x_n))$ converges to $f(x)$. What happens if the sequence (f_n) converges to f pointwise?

2. If a sequence (a_n) converges to a , show that the sequence (σ_n) defined by

$$\sigma_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n) \quad (n = 1, 2, \dots)$$

also converges to a .

3. For a given sequence (a_n) of real numbers, define

$$f(x) = \sum_{n=1}^{\infty} a_n g(n(x-n)) \quad (x \in \mathbb{R}),$$

where $g(x) = \max\{1 - |x|, 0\}$. Show that f is uniformly continuous on \mathbb{R} if and only if $\lim_{n \rightarrow \infty} a_n = 0$.

4. Find the Fourier series of the function $f(x) = \frac{\pi - x}{2}$ on the interval $[0, 2\pi]$. Show that this series converges uniformly on the interval $[\delta, 2\pi - \delta]$ for each $\delta \in (0, \pi)$. Finally, use this series to find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

5. For a continuous function $f : [-1, 1] \rightarrow \mathbb{R}$, show that

$$\lim_{n \rightarrow \infty} \int_{-1}^1 \frac{n}{2} e^{-n|x|} f(x) dx = f(0).$$

6. Let $f(z) = z^4 - 6z + 3$.

(a) Show that the equation $f(z) = 0$ has 4 roots in the disk $|z| < 2$.

(b) Evaluate

$$\int_{|z|=2} \frac{z^2 f'(z) dz}{f(z)}.$$

7. Evaluate the following integrals.

(a) $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx.$

(b) $\int_0^{\infty} \frac{\cos x}{x^2 + 1} dx.$

8. If $f : \Omega \rightarrow \mathbb{C}$ is holomorphic, $z_0 \in \Omega$, and $f'(z_0) = 0$, prove that f is not one-to-one in any neighborhood of z_0 .
9. Let Ω be an open subset of the complex plane. Suppose f_1, f_2, \dots are analytic in Ω and the sequence (f_j) converges to f uniformly on compact subsets of Ω . Prove the following:
- (a) f is analytic.
- (b) (f'_j) converges to f' uniformly on compact subsets of Ω .

10. Suppose $f \in H(\Omega)$, Ω contains the closed unit disc, and $|f(z)| < 1$ if $|z| = 1$. How many fixed points must f have in the disk? That is, how many solutions does the equation $f(z) = z$ have there?
11. In each case, find a conformal mapping of Ω onto the unit disk $|w| < 1$.
- (a) $\Omega = \{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$.
 - (b) $\Omega = \{z : 0 < \operatorname{Im} z < 1\}$.
 - (c) $\Omega = \{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0, |z| < 1\}$.
 - (d) $\Omega = \{z : \operatorname{Re} z > 0, 0 < \operatorname{Im} z < 1\}$.

12. Let f be an entire function and suppose there are positive constants M, R and n such that

$$|f(z)| \leq M|z|^n \quad (|z| \geq R).$$

Show that f is a polynomial of degree $\leq n$.

13. Suppose f is an entire function and g is a continuous function defined on a closed interval $[a, b]$. Show that $\int_a^b f(z)g(t)dt$ is an entire function of z .

14. Let U denote the unit disk in the complex plane. Suppose $f : U \rightarrow U$ is holomorphic and $f(0) = 0$.

- (a) Show that $\sum_{n=1}^{\infty} f(z^n)$ converges uniformly on compact sets in U .

- (b) Let $f_1 = f$ and $f_{n+1} = f \circ f_n$ for $n = 1, 2, \dots$. If $|f'(0)| < 1$, show that $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets in U .

15.

- (a) Let $\Omega = (\mathbb{C} \setminus \mathbb{Z}) \cup \{0\}$. Show that

$$\sum_{n=1}^{\infty} \frac{2z^2}{z^2 - n^2}$$

converges uniformly on compact sets in Ω .

- (b) It is known that

$$\pi z \cot \pi z = 1 + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 - n^2} \quad (z \in \Omega).$$

Use this fact to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(Hint: Find the first few terms of the Maclaurin series of $z \cot z$ and $\frac{2z^2}{z^2 - n^2}$.)

16. Show that an entire function f is a polynomial if and only if

$$\lim_{|z| \rightarrow \infty} |f(z)| = \infty.$$

Find all entire functions that are one-to-one.

17. In each case, find the Fourier integral $\int_{-\infty}^{\infty} f(x)e^{2\pi i t x} dx$.

- (a) $f(x) = e^{-\pi x^2}$.
- (b) $f(x) = \frac{1}{\cosh \pi x}$.
- (c) $f(x) = \frac{1}{1 + \pi^2 x^2}$.

(d) $f(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2$.

18. If $E \subset (0, 1)$ is Lebesgue measurable, $f : (0, 1) \rightarrow \mathbb{R}$ is of class C^1 and strictly increasing, show that $f(E)$ is Lebesgue measurable. Show that there are a Lebesgue measurable set $E \subset (0, 1)$ and a continuous non-decreasing function $f : (0, 1) \rightarrow \mathbb{R}$ such that $f(E)$ is not Lebesgue measurable.

19. If $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are integrable functions, define $f * g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y)dy$$

if $f(x - y)g(y)$ is an integrable function of y ; otherwise let $(f * g)(x) = 0$.

- (a) Show that for almost every x , $f(x - y)g(y)$ is an integrable function of y . (You may assume without proof that $f(x - y)$ is a measurable function on \mathbb{R}^2).
- (b) Show that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- (c) Show that $f * g = g * f$.
- (d) Show that $(f * g) * h = f * (g * h)$.
20. If $h : \mathbb{R} \rightarrow \mathbb{R}$ is of class C^1 , vanishes outside a bounded interval, and $f \in L^1(\mathbb{R})$, show that $h * f \in L^1(\mathbb{R})$, $h * f$ is of class C^1 , and $(h * f)' = h' * f$. Show that for every $f \in L^1(\mathbb{R})$ there is a sequence (f_n) of C^∞ functions such that $f_n \in L^1(\mathbb{R})$ for all n and $\|f_n - f\|_1 \rightarrow 0$ as $n \rightarrow \infty$.

21. Let ϕ be a nonnegative continuous function on \mathbb{R} with compact support such that $\int_{\mathbb{R}} \phi(x)dx = 1$, and let $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right)$.

- (a) Prove that if $1 \leq p < \infty$ and if $g \in L^p(\mathbb{R})$, then

$$\lim_{\varepsilon \rightarrow 0} \|\phi_\varepsilon * g - g\|_p = 0.$$

- (b) Is it true for $p = \infty$ in (a)? If not, provide a counterexample.

22. Suppose f is a measurable function on X , and μ a finite positive measure on X . Show that

$$\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$

23. Show that

$$\lim_{n \rightarrow \infty} \int_0^1 e^{inx^2} dx = 0.$$

More generally, if P is a non-constant real polynomial, show that

$$\lim_{n \rightarrow \infty} \int_0^1 e^{inP(x)} dx = 0.$$

24. If $f \in L^1(\mathbb{R})$, $f \geq 0$, and $\int_{\mathbb{R}} f(x) dx = 1$, show that

$$\left| \int_{\mathbb{R}} f(x)e^{-ix} dx \right| < 1.$$

25. If f is Lebesgue measurable on $[0, 1]$, show that $f \in L^2[0, 1]$ if and only if $f \in L^1[0, 1]$ and there is a monotone increasing function g such that for all closed intervals $[a, b]$ in $[0, 1]$,

$$\left| \int_a^b f(x)dx \right|^2 \leq (g(b) - g(a))|b - a|.$$

26. If μ is a Borel measure on $[0, 1]$ and $\int x^n d\mu(x) = 0$ for all $n \geq 0$, show that $\mu = 0$.
27. Suppose that $f(z)$ is analytic in $D(0; 1) = \{z : |z| < 1\}$, $f(0) = 1$, and $\operatorname{Re} f(z) > 0$ for all $z \in D(0; 1)$. Show that

$$\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|} \quad (z \in D(0; 1)).$$

28. An entire function f is said to be of *exponential type* if there are positive constants C and A such that

$$|f(z)| \leq Ce^{A|z|} \quad (z \in \mathbb{C}).$$

Show that an entire function f is of exponential type if and only if

$$\overline{\lim}_{n \rightarrow \infty} |f^{(n)}(0)|^{1/n} < \infty.$$

(Hint: $n! \sim (n/e)^n \sqrt{2\pi n}$ for large n).

29. Suppose that $f \in L^1(\mathbb{R})$ is non-negative. For $t \geq 0$, define

$$\phi(t) = m(\{x \in \mathbb{R} : f(x) > t\}).$$

Here m denotes the Lebesgue measure on \mathbb{R} . Prove the following identity:

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} \phi(t) dt.$$

30. Suppose f_n is a sequence of entire functions.

(a) Show that

$$f_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\rho e^{it} + z}{\rho e^{it} - z} \operatorname{Re} f_n(\rho e^{it}) dt + i \operatorname{Im} f_n(0) \quad (n = 1, 2, \dots; |z| < \rho).$$

(b) If $\{\operatorname{Re} f_n\}$ converges uniformly on compact subsets of the complex plane, and $\{f_n(0)\}$ converges, then show that $\{f_n\}$ converges uniformly on compact subsets of the complex plane.

31. For given $f \in L^1(\mathbb{R})$, define the Fourier transform of f as follows.

$$\hat{f}(\xi) \equiv \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x) dx.$$

(a) Show that if $f = \chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$, then

$$\hat{f}(\xi) = \begin{cases} b - a, & \text{if } \xi = 0, \\ \frac{\sin \pi(b-a)\xi}{\pi\xi} e^{-i\pi(a+b)\xi}, & \text{if } \xi \neq 0. \end{cases}$$

(b) Prove that the Fourier transform $\hat{\cdot} : L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ is a continuous linear operator with $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$.

(c) Show that $\lim_{|\xi| \rightarrow \infty} |\hat{f}(\xi)| = 0$.

32. Let f be a measurable function defined on the interval $[0, 1]$. Suppose that

$$\left| \int f(x) \chi_A(x) dx \right| \leq C m(A)^{1-\alpha}, \quad 0 < \alpha < 1$$

for any measurable set $A \subset [0, 1]$. Here $m(A)$ denotes the Lebesgue measure of A .

(a) Show for $\lambda > 0$ that

$$m(\{x \in [0, 1] : |f(x)| > \lambda\}) \leq (C/\lambda)^{\frac{1}{\alpha}}.$$

(b) Prove that $f \in L^p([0, 1])$ if $1 \leq p < 1/\alpha$.

(Hint: $\|f\|_{L^p([0,1])}^p = p \int_0^\infty \lambda^{p-1} m(\{x \in [0, 1] : |f(x)| > \lambda\}) d\lambda$).

33. Find a meromorphic function with simple poles at $\sqrt{n}, n = 0, 1, 2, \dots$ with residue 1. Express as an infinite series that converges uniformly on compact sets after omitting finite number of terms. Justify your answer.

34. Consider $(-z)^{s-1}$ defined on the complement of the positive real axis as $e^{(s-1)\log(-z)}$ with $-\pi < \text{Im} \log(-z) < \pi$. Let N be a positive integer and γ be a contour that consists of a positively oriented square with vertices at $\pm\pi(2N + \frac{1}{4}) \pm i\pi(2N + \frac{1}{4})$ and line segment parallel to the positive real axis and circle of radius ρ centered at the origin, and then the same line segment backwards as shown in figure 1: Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{(-z)^{s-1}}{e^z - 1} dz = \pi^{s-1} \sin \frac{\pi s}{2} \sum_{k=1}^N k^{s-1}.$$

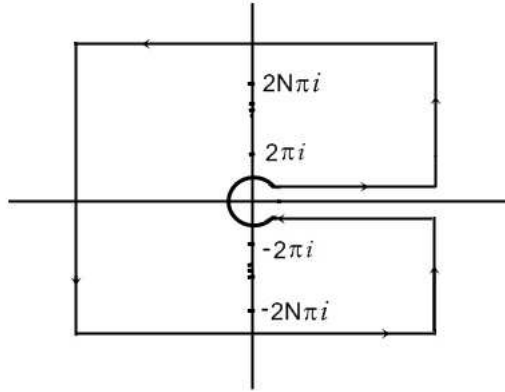


Figure 1: Schematic diagram of contour.